Ramanujan—For Lowbrows

Bruce C. Berndt and S. Bhargava

"No, Inspector," he said. "It is not at all like that, I am assuring you. You see, for a person of my sort—and I admit that we are a rare breed—numbers are so much in our minds there is hardly any question of writing them down, let alone adding one to another."...

"Let me give you one instance," he said. "Before I was beginning work just now, I was taking a short stroll, and I happened to see a handcartwalla. Now, being the sort of chap I am, I of course notice the number burned on the side of the cart: seventeen-twenty-nine. Now, does that mean anything to you yourself?"

"It is the number on the cart," Ghote answered guardedly. "By law it must be there."

Raghu Barde smiled his warm smile again.

"Ah, yes, the police view. But what do you think those figures meant to me? You would never guess. But the moment I was seeing them I said: Aha, the smallest number expressible as a sum of two cubes in two different ways. And, you know, if ever I am getting to marry, I suppose I will want a wife whose birth date comes to some number pleasing to me like that."

"I see," Ghote said.

And, although the mumbo jumbo about cubes and expressible meant nothing to him, and he could not help thinking that to choose a wife by number would be a much riskier proceeding than to let the astrologers choose one for you, he did dimly see what a different sort of life Raghu Barde lived from that of the common number-unencumbered man.

H. R. F. Keating
Dead on Time

1. INTRODUCTION. To celebrate the centenary of Ramanujan's birth, in June, 1987, an international conference was held at The University of Illinois at Urbana-Champaign [1]. Numerous roads through varied scenery brought researchers from Ramanujan's papers, problems, letters, notebooks, and unpublished manuscripts to a panoply of areas of contemporary research, including partitions, mock theta-functions, statistical mechanics, Lie algebras, probabilistic number theory, modular forms, elliptic functions, complex multiplication, hypergeometric series, q-series, asymptotic expansions, and beta integrals. Very few mathematicians have ever had such a broad impact on mathematical research. Although many results presented at the conference could be understood and appreciated by mathematicians outside these areas of research, this was a conference for *highbrows*.

Many of Ramanujan's beautiful discoveries, however, are easily understood, are elementary, and appeal to a wide variety of tastes. Thus, this paper is written for *lowbrows*. Only elementary algebra is needed to prove the lion's share of theorems reported here. Most are found in the unorganized portion of Ramanujan's second notebook, his third notebook, and problems that he posed for readers of the *Journal of the Indian Mathematical Society*. The results we describe fall under the headings of elementary algebra, equal sums of powers, and elementary number theory.

We begin our expedition in a taxi-cab as we recount G. H. Hardy's riding in taxi-cab no. 1729 to visit Ramanujan while lying ill in Putney. Some historical remarks are offered on the two representations $1^3 + 12^3 = 9^3 + 10^3$ of 1729. This leads us to Euler's solution, rediscovered by Ramanujan in a simpler form, of the diophantine equation $A^3 + B^3 = C^3 + D^3$.

We turn from equal sums of third powers to equal sums of fourth powers and ask "Did Ramanujan ever read Mathematical Magazine?" No, we are not speaking of the journal, Mathematics Magazine, published by the MAA, with the first issue appearing under a slightly different title in 1926, six years after Ramanujan's death. Some historical remarks will be made about Mathemati*cal* Magazine.

We next temporarily stop our journey to view what the authors consider to be one of the most captivating, enthralling finite identities in all of mathematics. Is this marvelous identity simply an accident on the road to sums of powers? Or are we at the base of the Himalayas—facing away from the mountains?

We next encounter three types of systems of equations. The first system leads us to sequences that decrease for a while, then increase for a while, etc. We must have roamed to a college campus, for these sequences involve radicals, infinitely many of them. Like most radicals, these have interesting properties. The second system leads us to a visit with S. Ramanujam. No, that is not a misprint! Is he really Ramanujan, or is he someone else? Our third system was solved beautifully by Ramanujan in his third published paper, but he did not realize that J. J. Sylvester had solved this system in 1851, nor was Ramanujan aware of the implications of his work. We provide a sketch of Ramanujan's clever proof.

Proceeding from a sketch to a complete landscape, we provide proofs of some interesting properties of roots of cubic polynomials that Ramanujan discovered. As applications, we offer two curious trigonometric identities.

For our last proof, we establish sharp bounds for a sum giving the largest power of a prime dividing $n!$.

We conclude our paper with some approximations to π.

Several references will be made to Ramanujan's notebooks [**26**], published in two volumes. The second volume contains the second and third notebooks, and all page numbers in this paper refer to the pagination in this volume.

2. SUMS OF POWERS. Many readers are familiar with the famous taxi-cab story immortalized by Hardy [**27**, p. xxxv]. "I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab no. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.'" (It is clear that the author of the opening passage about a handcart with 1729 imprinted on its side was acquainted with this delightful incident in the life of Ramanujan and Hardy. A handcartwalla is a person who pulls a two-wheeled handcart, normally carrying one or two people, and is no longer a common sight in present day India. The suffix "walla" comes from Hindi.) In fact, Ramanujan had previously recorded these two representations for 1729, $1^3 + 12^3$ and $9^3 + 10^3$, on page 225 of his second notebook [**26**]. However, this example appears to have been first noticed by B. Frenicle de Bessy in 1657. Frénicle and J. Wallis each found additional examples for two equal sums of two cubes. A bitter argument ensued with each accusing the other of using trivial methods. Since P. Fermat also frequently was feuding with these two men, letters detailing their acrimony can be

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The frontispiece of a volume of Wordsworth's poetry. The volume was awarded to the young Ramanujan for his "outstanding work in Maths." Such prizes for mathematical contests were common in Ramanujan's hometown, Kumbakonan, and throughout India of the period.

found in Fermat's *Oeuvres* [**11**, pp. 419–420; 427–457] and E. T. Bell's book [**2**, Chapter 12], as well as in L. E. Dickson's *History* [**8**, p. 552]. In 1898, C. Moreau [**18**] found the ten solutions of $A^3 + B^3 = C^3 + D^3$ with the sums less than 100,000. After 1729, the next largest sum is $4104 = 2^3 + 16^3 = 9^3 + 15^3$.

From another viewpoint, Ramanujan provided Hardy with solutions to the classical diophantine equation

$$A^3 + B^3 + C^3 = D^3. \tag{2.1}$$

L. Euler [**10**] completely solved (2.1) for positive or negative rational solutions. At three places in his notebooks, Ramanujan addresses the problem of finding solutions of (2.1). In Entry 20(iii) of Chapter 18 and on page 266 in the unorganized portion of his second notebook, Ramanujan provides parametric solutions to (2.1), but they are not as general as Euler's. But near the end of his third notebook [**26**, p. 387], Ramanujan offers a family of solutions equivalent to Euler's general solution. Both Hardy [**13**, p. 11] and G. N. Watson [**30**] discussed one of Ramanujan's less general solutions to (2.1). They had no knowledge of Ramanujan's general solution, because they did not have access to the third notebook. We quote Ramanujan's theorem.

Theorem. *If*

$$\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2,$$

then

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3. \tag{2.2}$$

As an example, we recover the two pairs of aforementioned taxi-cab cubes by putting $(\alpha, \beta, \gamma, \lambda) = (3, 0, 1, 3)$ in (2.2).

Although several formulations equivalent to Euler's general solution have been discovered, Ramanujan's formulation (2.2) appears to be the simplest of all. The problem of completely characterizing all positive integral solutions of (2.1) is unsolved.

On the other hand, Euler conjectured that there were no positive integral solutions to

$$A^4 + B^4 + C^4 = D^4.$$

It was not until 1988 that Euler's conjecture was shown to be false by N. D. Elkies [**9**], who found an infinite class of solutions.

Ramanujan derived several theorems providing infinite families of solutions for equal sums of powers. For example, toward the end of this third notebook [**26**, p. 384], he writes two parametric solutions for representing a fourth power as a sum of five fourth powers.

Theorem. *If s, t, m, and n are arbitrary, then*

$$(8s^2 + 40st - 24t^2)^4 + (6s^2 - 44st - 18t^2)^4 + (14s^2 - 4st - 42t^2)^4 + (9s^2 + 27t^2)^4 + (4s^2 + 12t^2)^4 = (15s^2 + 45t^2)^4 \tag{2.3}$$

and

$$(4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 = (5m^2 + 15n^2)^4. \tag{2.4}$$

Ramanujan recorded several examples. For instance, if we set $s = 1$ and $t = 0$ in (2.3), we find that

$$4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4.$$

Formula (2.3) is due to C. B. Haldeman [**12**, pp. 289–290] in 1904. Uncannily, Ramanujan used the same notation and recorded the terms in the same order as Haldeman! Likewise, (2.4) was established by Haldeman [**12**, p. 289] and slightly later by A. Martin [**15**, pp. 325–326, 331]. Ramanujan does not use Haldeman's notation in (2.4) but does employ Martin's notation!

Ramanujan recorded his results in notebooks from about 1903 until he departed for England in 1914. The 16 chapters in the first notebook and the 21 chapters in the second evince a progressive maturation from more elementary mathematics to much deeper results. The third notebook, however, contains both very elementary results as well as advanced results. While the latter theorems may have been recorded in Cambridge, the former results were probably recorded early in the period 1903–1914. Since in India Ramanujan did not have access to even the primary mathematical journals of his day, it is extremely unlikely that he could have seen the obscure journal, *Mathematical Magazine*, in which Martin and Haldeman published their results. Thus, the notation in (2.3) and (2.4) being identical with that of Haldeman and Martin, respectively, must be coincidental.

Mathematical Magazine was founded and edited by Martin and was devoted to "elementary mathematics." Issues of the first volume were published quarterly in

1882–1884 at a cost of 50 cents per issue or one dollar per year. The second and last volume of 12 issues was published over the years 1890–1904, with the last four issues appearing in January, 1895; January, 1896; December, 1898; and January, 1904. The last issue contains four papers, three by Martin and one by Haldeman. In the penultimate issue, under the heading "Editorial Items," we learn that "Since No. 10 of the Magazine was published, three able contributors have 'crossed over' and 'passed beyond the confines of earth.'" It is likely that an even greater number "crossed over" between the 11th and 12th issues. Possibly due to complaints registered by readers disgruntled over the irregularity at which issues appeared, the price per issue had dropped to 30 cents.

Toward the end of the third notebook [**26**, p. 386], Ramanujan records one of the most fascinating identities we have ever seen.

Theorem. *Let a, b, c, and d denote any numbers such that $ad = bc$. Then*

$$\begin{aligned}64\{(a+b+c)^6 &+ (b+c+d)^6 - (c+d+a)^6 - (d+a+b)^6 \\ &+ (a-d)^6 - (b-c)^6\} \\ \times\{(a+b+c)^{10} &+ (b+c+d)^{10} - (c+d+a)^{10} - (d+a+b)^{10} \\ &+ (a-d)^{10} - (b-c)^{10}\} \\ = 45\{(a+b+c)^8 &+ (b+c+d)^8 - (c+d+a)^8 - (d+a+b)^8 \\ &+ (a-d)^8 - (b-c)^8\}^2. \end{aligned} \tag{2.5}$$

The hypothesis $ad = bc$ was omitted by Ramanujan, although it does appear as a hypothesis for some related results on the previous page.

We first transcribe (2.5) into a somewhat more transparent form. For each positive integer m, set

$$\begin{aligned}F_{2m}(a,b,c,d) = (a+b+c)^{2m} &+ (b+c+d)^{2m} - (c+d+a)^{2m} \\ &-(d+a+b)^{2m} + (a-d)^{2m} - (b-c)^{2m}.\end{aligned}$$

Put $b = ax$, $c = ay$, and $d = axy$, which does not contravene the hypothesis $ad = bc$. Then it is easy to see that

$$F_{2m}(a,b,c,d) = a^{2m} f_{2m}(x,y),$$

where

$$\begin{aligned}f_{2m}(x,y) = (1+x+y)^{2m} &+ (x+y+xy)^{2m} - (y+xy+1)^{2m} \\ &-(xy+1+x)^{2m} + (1-xy)^{2m} - (x-y)^{2m}.\end{aligned} \tag{2.6}$$

Hence, (2.5) can be put in the form

$$64 f_6(x,y) f_{10}(x,y) = 45 f_8^2(x,y). \tag{2.7}$$

We first employed the computer algebra system *Mathematica* to verify (2.7). Next, using Mathematica, we attempted to find other identities like (2.7) involving $f_{2m}(x,y)$ for $m \le 10$, but we were unsuccessful. We fortunately found a much more informative proof of (2.7) that is not merely a verification via computer algebra [**6**]. We will not repeat that proof here but instead offer a few additional remarks.

By inspection, we easily see that $x = 0, 1, -1, -2, -1/2$ are zeros of $f_{2m}(x, y)$. By symmetry, $y = 0, 1, -1, -2, -1/2$ are also zeros. Since f_{2m} has degree (at most) $2m$ in each of the variables x and y, it follows that $f_2(x, y) \equiv 0 \equiv f_4(x, y)$. In our original notation, we have therefore proved that, if $ad = bc$, then

$$(a + b + c)^n + (b + c + d)^n + (a - d)^n = (c + d + a)^n + (d + a + b)^n + (b - c)^n, \tag{2.8}$$

where $n = 2$ or 4. These are the aforementioned results that appear on page 385 of [**26**]. We have therefore returned to the problem of generating equal sums of biquadrates. Although many results have appeared in the literature yielding two equal sums of three biquadrates [**8**, pp. 653–657], none appear as simple as Ramanujan's identity (2.8).

Are (2.5) and (2.7) merely accidents, or are they a manifestation of some far deeper theorem?

3. ELEMENTARY ALGEBRA. In courses and texts on beginning calculus, students encounter many monotonic sequences in their study of sequences and series. An inquisitive student may ask for naturally occurring examples of sequences that increase for a while, then decrease for a while, etc. As we shall see, some infinite sequences of nested radicals of Ramanujan provide excellent examples.

Mrs. Ramanujan (S. Janaki Ammal) and W. Narayanan, one of her two adopted sons.

In 1914, Ramanujan [**22**], [**27**, pp. 327–329] posed the following problem to readers of the *Journal of the Indian Mathematical Society*: Solve completely

$$x^2 = y + a, \quad y^2 = z + a, \quad \text{and} \quad z^2 = x + a. \tag{3.1}$$

Concomitantly, he asked for the evaluation of three infinite sequences of nested radicals. Toward the end of his second notebook [**26**, pp. 305–307], Ramanujan recorded further and more general results. It is not difficult to see that x is a root of an octic polynomial. This polynomial can be factored over the quadratic field

$Q(\sqrt{4a-7})$ into one quadratic and two cubic factors. These factors are correctly given by Ramanujan in his solution [**22**], but the factors given in the solution printed in his *Collected Papers* [**27**, pp. 327–329] contain four sign errors.

From the equalities (3.1), we find that

$$x = \sqrt{a+y} = \sqrt{a+\sqrt{a+z}} = \sqrt{a+\sqrt{a+\sqrt{a+x}}}$$
$$= \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}\,. \tag{3.2}$$

Each square root should be considered two-valued, and so we are led to eight infinite sequences of nested radicals corresponding to the eight roots of our octic polynomial. First, we should determine those values of a for which the infinite radical in (3.2) converges. This is not an easy problem, but each of the eight sequences in (3.2) converges at least for $a \geq 2$ [**5**, Chapter 22]. As a specific example, let

$$a_1 = \sqrt{a}, \qquad a_2 = \sqrt{a-\sqrt{a}}, \qquad a_3 = \sqrt{a-\sqrt{a+\sqrt{a}}},$$
$$a_4 = \sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a}}}}, \ldots,$$

where the sequence of signs $-,+,+,\ldots$ appearing in the nested radicals has period 3. A careful analysis shows that

$$a_{6n+1} > a_{6n+2} > a_{6n+3} > a_{6n+4}$$

and

$$a_{6n+4} < a_{6n+5} < a_{6n+6} < a_{6n+7},$$

for each nonnegative integer n. Furthermore,

$$0 < a_4 < a_{10} < \cdots < a_{6n+4} < a_{6n+7} < a_{6n+1} < \cdots < a_7 < a_1 = \sqrt{a}\,.$$

Thus, $\{a_{6n+1}\}$ and $\{a_{6n+4}\}$ converge. Next, it must be shown that $\{a_{3n+1}\}$ converges and, lastly, that $\{a_n\}$ converges. The details in this analysis are not easy [**5**, Chapter 22].

If we solve the two cubic equations mentioned above, it is not easy, in general, to identify the roots with the appropriate infinite sequences of radicals. For example,

$$\lim_{n\to\infty} a_n = \frac{A-1}{6} + \frac{2}{3}\sqrt{4a+A}\,\sin\left(\frac{1}{3}\arctan\frac{2A+1}{3\sqrt{3}}\right), \tag{3.3}$$

where $A = \sqrt{4a-7}$. We made these identifications by expanding both the algebraically determined roots and the infinite radicals around "$a = \infty$." For example, both sides of (3.3) have the asymptotic expansions

$$\sqrt{a} - \frac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{1}{4a} + \cdots,$$

as a tends to ∞. For particular numerical examples, the proper identifications are easier to make. For instance, if $a = 2$ in (3.3),

$$2\sin\left(\frac{\pi}{18}\right) = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\cdots}}}}\,.$$

Later, Ramanujan [25], [27, p. 332] submitted the similar problem of determining the simultaneous solutions of the system,

$$x^2 = a + y, \qquad y^2 = a + z, \qquad z^2 = a + u, \quad \text{and} \quad u^2 = a + x,$$

to the *Journal of the Indian Mathematical Society*. Fourteen years elapsed before a solution by G. N. Watson [29] was published, while another solution can be found in [5, Chapter 22]. As above, interesting sequences of nested radicals arise. For example,

$$\frac{1}{2}\left(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}\right) = \sqrt{5 + \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 + \cdots}}}}},$$

where the infinite sequence of signs $+, +, -, +, \cdots$ has period 4.

The theory of infinite sequences of nested radicals has not been well developed, probably because general theorems are difficult to obtain and convergence is slow. For further examples, theorems, and references to the literature, see [3, pp. 108–112] and [5, Chapter 22].

In the unorganized portions of his notebooks [26] and in the problem sections of the *Journal of the Indian Mathematical Society*, Ramanujan offers other problems on systems of equations. Thus, on page 338 of [26], he asks for the solutions of

$$\frac{x^5 - a}{x^2 - y} = \frac{y^5 - b}{y^2 - x} = 5(xy - 1),$$

where a and b are arbitrary constants. There are 25 pairs (x, y) of solutions. The special case $a = 6$, $b = 9$ appeared as Question 284 [20], [27, pp. 322–323] in the *Journal of the Indian Mathematical Society*. Ramanujan's solution was the only one received, and a similar solution to the more general problem can be found in [5, Chapter 22].

Question 284 was the fourth problem that Ramanujan published in the *Journal of the Indian Mathematical Society*. The first five problems that Ramanujan posed to *Journal* readers were published under the name S. Ramanujam. Ramanujan and Ramanujam are two versions of the same Sanskrit name RAMANUJAHA, which means younger brother of Rama.

We mention one further system of equations studied by Ramanujan. On page 338 of his second notebook, Ramanujan asks, in slightly different notation, for the solutions of the system of $2n$ equations,

$$x_1 y_1^{j-1} + x_2 y_2^{j-1} + \cdots + x_n y_n^{j-1} = a_j, \qquad 1 \le j \le 2n, \tag{3.4}$$

where $x_1, \dots, x_n, y_1, \dots, y_n$ are $2n$ unknowns, and in his short paper [21], [27, pp. 18–19], Ramanujan presents his clever solution, which we briefly indicate.

Ramanujan defines

$$\varphi(\theta) := \sum_{j=1}^{n} \frac{x_j}{1 - \theta y_j}. \tag{3.5}$$

When $\varphi(\theta)$ is expanded in a power series in θ, it is seen that the coefficient of θ^k is a_{k+1}, $0 \le k \le 2n - 1$. On the other hand, $\varphi(\theta)$ has the form

$$\varphi(\theta) = \frac{\sum_{j=0}^{n-1} A_{j+1}\theta^j}{1 + \sum_{j=1}^{n} B_j \theta^j}. \tag{3.6}$$

Clearing the denominator in (3.6) and using the aforementioned power series for $\varphi(\theta)$, we can determine first the coefficients B_j, $1 \le j \le n$, and secondly the coefficients A_j, $1 \le j \le n$, in terms of $a_1, a_2, \ldots, a_{2n}$ by equating coefficients of like powers of θ. Having explicitly determined A_j and B_j, $1 \le j \le n$, we substitute these values into (3.6) and once again expand $\varphi(\theta)$ into partial fractions. Comparing the result with (3.5), we determine x_j and y_j, $1 \le j \le n$.

It is easy to see that the system (3.4) is equivalent to the single equation

$$\sum_{i=1}^{n} x_i (y_i s + t)^{2n-1} = \sum_{j=0}^{2n-1} \binom{2n-1}{j} a_{j+1} s^j t^{2n-1-j}.$$

Thus, Ramanujan's query is equivalent to the question: When can a binary $(2n-1)-ic$ form be represented as a sum of n $(2n-1)th$ powers? In 1851, Sylvester [**28**, pp. 203–216, 265–283] found the following necessary and sufficient conditions for a solution: The system of n equations

$$a_j u_1 + a_{j+1} u_2 + \cdots + a_{j+n} u_{n+1} = 0, \qquad 1 \le j \le n,$$

must have a solution $u_1, u_2, \ldots, u_{n+1}$ such that the $n-ic$ form

$$p(w,z) := \sum_{j=0}^{n} u_{j+1} w^j z^{n-j}$$

can be represented as a product of n distinct linear forms. This is true for a general $2n$-tuple $(a_1, a_2, \ldots, a_{2n})$ in the sense of algebraic geometry. Thus, the numbers $y_1, y_2, \ldots, y_n$ are related to the factorization of $p(w,z)$. Sylvester's theorem belongs to the subject of invariant theory, which was developed in the late 19th and early 20th centuries. For a contemporary treatment, but with classical language, see a paper by J. P. S. Kung and G.-C. Rota [**14**].

We next consider the following theorem of Ramanujan [**26**, p. 325].

Theorem. *Let* α, β, *and* γ *denote the roots of the cubic equation*

$$x^3 - ax^2 + bx - 1 = 0. \tag{3.7}$$

Then, for a suitable determination of roots,

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} = (a + 6 + 3t)^{1/3} \tag{3.8}$$

and

$$(\alpha\beta)^{1/3} + (\beta\gamma)^{1/3} + (\gamma\alpha)^{1/3} = (b + 6 + 3t)^{1/3}, \tag{3.9}$$

where

$$t^3 - 3(a + b + 3)t - (ab + 6(a+b) + 9) = 0. \tag{3.10}$$

Since this beautiful elementary theorem is evidently new and since a short proof can be given, we provide one here.

Proof: Noting, from (3.7), that $\alpha\beta\gamma = 1$, let

$$z^3 - \theta z^2 + \varphi z - 1 = 0 \tag{3.11}$$

denote the cubic polynomial with roots $\alpha^{1/3}$, $\beta^{1/3}$, and $\gamma^{1/3}$, chosen so that their product equals 1. Cubing both sides of the equality

$$z^3 - 1 = \theta z^2 - \varphi z,$$

we find that

$$(z^3 - 1)^3 - \theta^3 z^6 + \varphi^3 z^3 + 3\theta\varphi z^3 (z^3 - 1) = 0. \tag{3.12}$$

Since $\alpha^{1/3}$, $\beta^{1/3}$, and $\gamma^{1/3}$ are roots of (3.11), they are also roots of (3.12). As a cubic polynomial in z^3, (3.12) thus has the roots α, β, and γ.

Comparing (3.7) and (3.12), we deduce that

$$a = \theta^3 + 3 - 3\theta\varphi \tag{3.13}$$

and

$$b = \varphi^3 + 3 - 3\theta\varphi. \tag{3.14}$$

If we define t by

$$\theta^3 = a + 6 + 3t, \tag{3.15}$$

then, by (3.11) and (3.15),

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} = \theta = (a + 6 + 3t)^{1/3},$$

which proves (3.8). Also, by (3.13)–(3.15),

$$\varphi^3 = b - 3 + 3\theta\varphi = b + \theta^3 - a = b + 6 + 3t. \tag{3.16}$$

Hence, by (3.11) and (3.16), (3.9) is established. From (3.13) and (3.15),

$$3 + t = \theta\varphi. \tag{3.17}$$

Thus, by (3.15)–(3.17),

$$(3 + t)^3 = \theta^3\varphi^3 = (a + 6 + 3t)(b + 6 + 3t).$$

Expanding both sides, collecting terms, and simplifying, we deduce (3.10).

On page 356 of [**26**], the last page of the second notebook, Ramanujan offers the equalities

$$\left(\cos\frac{2\pi}{9}\right)^{1/3} + \left(\cos\frac{4\pi}{9}\right)^{1/3} - \left(\cos\frac{\pi}{9}\right)^{1/3} = \left\{\frac{3}{2}(9^{1/3} - 2)\right\}^{1/3} \tag{3.18}$$

and

$$\left(\sec\frac{2\pi}{9}\right)^{1/3} + \left(\sec\frac{4\pi}{9}\right)^{1/3} - \left(\sec\frac{\pi}{9}\right)^{1/3} = \{6(9^{1/3} - 1)\}^{1/3}, \tag{3.19}$$

which are applications of (3.8) and (3.9), respectively, with $a = 0$, $b = -3$, and $t = -9^{\frac{1}{3}}$. Equality (3.18) was posed as a problem by Ramanujan in the *Journal of the Indian Mathematical Society* [**23**], [**27**, p. 329]. Proofs of (3.18) and (3.19) can also be found in Berndt's book [**5**, Chapter 22].

4. NUMBER THEORY. Suppose p is a prime and n is a positive integer. Then, by a well-known theorem in elementary number theory [**19**, p. 182], the highest power of p dividing $n!$ equals

$$\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right] =: N.$$

Despite the widespread use of this theorem by number theorists for many years, the inequalities

$$\frac{n}{p-1} - \frac{\log(n+1)}{\log p} \leq N \leq \frac{n-1}{p-1}, \tag{4.1}$$

given by Ramanujan [**26**, p. 378] in his third notebook do not appear to have been heretofore noticed. Both inequalities in (4.1) are sharp. If $n = p^m$ for some positive integer m, an elementary calculation shows that $N = (n-1)/(p-1)$.

On the other hand, if $n = p^{m+1} - 1$, by a direct calculation with the observation that $m + 1 = \log(n + 1)/\log p$,

$$N = \frac{n}{p-1} - \frac{\log(n+1)}{\log p}.$$

In fact, Ramanujan stated (4.1) with p replaced by an arbitrary positive integer $a \geq 2$.

Bhargava, Adiga, and Somashekara [7] have given one proof of (4.1) when p is any positive integer exceeding 1. We offer another proof here.

Proof of (4.1): First, by writing n in base p, i.e., by setting

$$n = \sum_{j=0}^{m} b_j p^j, \qquad 0 \leq b_j \leq p - 1, \quad b_m \neq 0,$$

we find, after a straightforward calculation, that

$$N = \frac{n}{p-1} - \frac{1}{p-1} \sum_{j=0}^{m} b_j, \tag{4.2}$$

and so the second inequality in (4.1) follows.

The first inequality in (4.1) is more difficult to establish. We are very grateful to B. Reznick for supplying the following elegant proof.

Set

$$b = \sum_{j=0}^{m} b_j.$$

Then, by (4.2), it suffices to prove that

$$b \leq (p-1)\frac{\log(n+1)}{\log p}. \tag{4.3}$$

Write

$$b = k(p-1) + r, \qquad 0 \leq r \leq p - 2. \tag{4.4}$$

Then

$$\begin{aligned} n &\geq (p-1)p^0 + (p-1)p + (p-1)p^2 + \cdots + (p-1)p^{k-1} + rp^k \\ &= (r+1)p^k - 1. \end{aligned}$$

It follows that

$$\begin{aligned} (p-1)\frac{\log(n+1)}{\log p} &\geq (p-1)\frac{\log((r+1)p^k)}{\log p} \\ &= k(p-1) + (p-1)\frac{\log(r+1)}{\log p}. \end{aligned} \tag{4.5}$$

By (4.3)–(4.5), we shall be finished with the proof if we can show that

$$r \leq (p-1)\frac{\log(r+1)}{\log p}. \tag{4.6}$$

First, if $r = 0$, (4.6) clearly holds with equality.

If $r \geq 1$, (4.6) can be written in the form

$$\frac{r}{\log(r+1)} \leq \frac{p-1}{\log p},$$

or

$$f(r) \leq f(p-1), \tag{4.7}$$

where

$$f(x) := \frac{x}{\log(x+1)}.$$

However, by elementary calculus, $f(x)$ is strictly increasing for positive integral x. Since $1 \leq r \leq p-2$, (4.7) is therefore valid with a strict inequality, and so the proof is complete.

As remarked in the Introduction, we conclude this short sampling of Ramanujan's elementary discoveries with a note on π. Continued fractions provide excellent rational approximations to π. Thus, the simple continued fraction

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{293} + \cdots$$

yields the successive approximations $\frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$. Note that

$$\frac{355}{113} = 3.14159\,29\ldots,$$

which agrees with the decimal expansion of $\pi = 3.14159\,26535\ldots$ through 6 decimal places. The appearance of a "large" fourth partial quotient, 293, is primarily responsible for this success.

Taking a brief diversion in his famous paper on approximations to π [**24**], [**27**, p. 35], Ramanujan offers the approximation

$$\pi \approx \left(97\frac{1}{2} - \frac{1}{11}\right)^{1/4} = 3.14159\,26526\ldots, \tag{4.8}$$

which "was obtained empirically." How did Ramanujan deduce this unusual approximation, which is also found in his second and third notebooks [**26**, pp. 217, 375]? N. D. Mermin [**16**], [**17**, pp. 304–305] has offered the best explanation for Ramanujan's approximation (4.8). In the decimal expansion of $\pi^4 = 97.409091034002\ldots$, observe that the pair of digits 09 appears twice in succession followed by the pair 10; which is 'close' to 09. Thus,

$$97.40909090909\ldots = \frac{2143}{22} = 97\frac{1}{2} - \frac{1}{11}$$

is a natural approximation to π^4.

Ramanujan's facility with continued fractions is unequaled in mathematical history, and so he might have observed that [**16**], [**17**], [**4**, p. 151]

$$\pi^4 = 97 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1} + \frac{1}{16539} + \frac{1}{1} + \cdots.$$

Truncating this continued fraction just before the "super large" partial quotient 16,539 gives the approximation (4.8).